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The Heat Equation with Stochastic Non-linear Boundary Conditions

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In this paper we examine the problem of the heat equation with non-linear boundary conditions of stochastic type.

INTRODUCTION

In this paper we consider a problem for the heat equation in which a stochastic relation is imposed between the interior values and the boundary values of the solution.

In earlier works [1–3] problems of this form appear in which deterministic relations are imposed. The problem which we study here is one in which the boundary values are dependent on the solution in the interior. Such problems are typically posed by a 'Thermostat' control unit whose performance varies with differing temperatures. In our case the assumption that the relation imposed between the interior values and the boundary values of the solution is of stochastic type means that our "Thermostat" has "noise." Moreover, we assume that our "Thermostat" is aging.

In Section 1 we introduce the notation to be used and the statement of the problem. Various lemmas necessary for our later work are proved in Section 2. In Section 3 we prove the existence of a solution and we study its asymptotic behaviour (for large time). The connection between the asymptotic behaviour of the solution in the stochastic and deterministic cases is examined in Section 4. In that section we also examine the "aging process" that takes place in the "Thermostat."

1. STATEMENT OF THE PROBLEM

We consider the heat equation

$$u_t = k(u_{xx} + u_{yy} + u_{zz}) \tag{1}$$

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for $(x, y, z) \in (R^+)^3$, $0 < t < \infty$, where $k > 0$ and $u = u(x, y, z, t)$, with the initial boundary conditions:

$$u(x, y, z, 0) = f(x, y, z),$$

$$u(0, y, z, t) = F(u(x_1, y_1, z_1, t), ..., u(x_m, y_m, z_m, t), y, z, t), \tag{3}$$

$$u(x, 0, z, t) = F(u(x_1, y_1, z_1, t), ..., u(x_m, y_m, z_m, t), x, z, t), \tag{4}$$

$$u(x, y, 0, t) = F(u(x_1, y_1, z_1, t), ..., u(x_m, y_m, z_m, t), x, y, t). \tag{5}$$

In earlier similar works [1–2] F was taken to be a given deterministic function. In this work we shall assume that

$$F \equiv F(\xi_1, \xi_2, ..., \xi_m, \eta_1, \eta_2, \eta_3), \qquad -\infty < \xi_i < \infty, \eta_j > 0,$$
$$i = 1, ..., m, j = 1, 2, 3,$$

represents a stochastic process; i.e., for all constant values of ξ_i and η_j, F is a random variable. We also assume that $f(x, y, z)$ is a random variable for each $x, y, z > 0$. F represents the operation of the "Thermostat." The system is monitored at the m fixed points (x_i, y_i, z_i), $i = 1, 2, ..., m$. The assumption that F is a random variable means that there is "noise" in the "Thermostat." The mathematical representation of the "aging" of F will be presented in Section 4.

We shall use the following definitions:

DEFINITION 1. We will let (Ω, A, P) denote a probability measure space where the non-empty set Ω is the sample space, A is a σ-algebra of subsets of Ω and P is a complete probability measure on A.

DEFINITION 2. The symbol $L_2 \equiv L_2(\Omega, A, P)$ will denote the collection of all random variables z such that $Ez^2 = \int_\Omega z^2 \, dP < \infty$. The space L_2 is a Banach space with the norm

$$\|z\|_{L_2} = \left(\int_\Omega z^2 \, dP \right)^{1/2}.$$

DEFINITION 3. The symbol $L_2^m \equiv L_2^m(\Omega, A, P)$ will denote the collection of all random variables $x = (x_1, x_2, ..., x_m)$ such that each coordinate x_i belongs to L_2. The space L_2^m is a Banach space with the norm defined by

$$\|x\|_{L_2^m} = \left(\sum_{i=1}^{m} Ex_i^2 \right)^{1/2}.$$

DEFINITION 4. The symbol $L_\infty \equiv L_\infty(\Omega, A, P)$ will denote the collection of all random variables z such that P ess sup $|z| < \infty$. That is, $\inf_{\Omega_0 \subset \Omega}(\sup_{\Omega - \Omega_0} |z|) < \infty$, where $P(\Omega_0) = 0$. We define the norm on L_∞ by

$$\|z\|_{L_\infty} = P \text{ ess sup } |z|.$$

Let R be the set of real numbers and denote by R^+ the set of nonnegative real numbers.

DEFINITION 5. $C \equiv C(R^+, L_2^m)$ will be the space of all vector functions x defined on R^+ with range contained in L_2^m such that

(a) x is mean square continuous, i.e.,

$$E \sum_{i=1}^{m} [x_i(t+h) - x_i(t)]^2 \to 0 \qquad \text{as} \quad h \to 0,$$

and

(b) x is uniformly bounded (m.s.), i.e.,

$$\sup_{t \geqslant 0} \left\{ E \sum_{i=1}^{m} x_i^2(t) \right\} < \infty.$$

The space C is a Banach space with the norm

$$\|x\|_C = \sup_{t \geqslant 0} \left\{ E \sum_{i=1}^{m} x_i^2(t) \right\}^{1/2}.$$

DEFINITION 6. We denote by $C((R^+)^4, L_2)$ the space of all random functions u defined on $(R^+)^4$ with values in L_2 which are mean square continuous with respect to each argument, i.e.,

$$E(u(x+h, y, z, t) - u(x, y, z, t))^2 \to 0, \qquad \text{as} \quad h \to 0,$$

with similar relations holding for y, z, t. We also denote by $C(R^m \times (R^+)^3, L_2)$ the space of all random functions defined on $R^m \times (R^+)^3$ with values in L_2 which are mean square continuous with respect to each argument.

We shall make the following assumptions on the random function F:

(A1) $F(\xi, \eta_1, \eta_2, t, \omega)$ belongs to $C(R^m \times (R^+)^3, L_2)$ and is continuous with respect to ξ a.e. in Ω. (Here ξ is an m-dimensional vector.)

(A2) $F(0, \eta_1, \eta_2, t, \omega) = 0$ a.e. in Ω. (Here 0 is the zero vector in R^m.)

(A3) F satisfies a Lipschitz condition of the form

$$|F(v, \eta_1 \eta_2, \eta_3, \omega) - F(u, s_1, s_2, s_3, \omega)| \leqslant \sum_{i=1}^{m} a_i(\omega)\,|v_i - u_i| + \sum_{j=1}^{3} b_j(\omega)\,|\eta_j - s_j|,$$

where $u, v \in R^m$ and $a_i(\omega)$, $b_j(\omega)$ are positive random variables belonging to $L_2(\Omega, A, P)$.

(A4) $a_i(\omega)$, $b_j(\omega)$ belong to L_∞.

We seek a function $u(x, y, z, t)$ such that

(I) $u(x, y, z, t) \in C((R^+)^4, L_2)$;

(II) there exists a stochastic function $u_t(x, y, z, t) \in C((R^+)^4, L_2)$ such that

$$E\left(\frac{u(x, y, z, t+h) - u(x, y, z, t)}{h} - u_t(x, y, z, t)\right)^2 \to 0 \quad \text{as } h \to 0;$$

(III) there exist stochastic functions u_{xx}, u_{yy}, $u_{zz} \in C((R^+)^4, L_2)$ defined as in II;

(IV) $u_t = k(u_{xx} + u_{yy} + u_{zz})$ a.e., in Ω;

(V) $u(x, y, z, 0) = f(x, y, z)\, f \in C((R^+)^3, L_2)$ and f is of exponential type [5, p. 54], i.e.,

$$h(x, y, z, t) = \int_0^\infty \int_0^\infty \int_0^\infty g(x, x', t)\, g(y, y', t)\, g(z, z', t) f(x', y', z')\, dx'\, dy'\, dz' \quad (6)$$

exists, where

$$g(x, x'\ t) = \frac{1}{2\sqrt{\pi k t}}\{e^{-(x-x')^2/4kt} - e^{-(x+x')^2/4kt}\}. \quad (7)$$

Here the integral is a stochastic mean square Riemann integral [4, p. 27].

(VI)

$$u(0, y, z, t) = F(u(x_1, y_1\, z_1, t), \ldots, u(x_m, y_m, z_m, t), y, z, t, \omega),$$
$$u(x, 0, z, t) = F(u(x_1, y_1, z_1, t), \ldots, u(x_m, y_m, z_m, t), x, z, t, \omega),$$
$$u(x, y, 0, t) = F(u(x_1, y_1, z_1, t), \ldots, u(x_m, y_m, z_m, t), x, y, t, \omega).$$

2. Basic Lemmas

In this section we introduce some basic lemmas that are necessary to give meaning to condition VI and which will also be used in the sequel.

Lemma 1. *Assme that F satisfies condition* A1 *and $y(t, \omega) = (y_1(t, \omega),..., y_m(t, \omega))$ is a vector of random variables. Then*

$$S_y(t, \omega) = F(y_1(t, \omega),..., y_m(t, \omega), \eta_1, \eta_2, t, \omega)$$

is a random variable for each $\eta_1, \eta_2, t \in R^+$.

Proof. Let $a_1,..., a_m$ be m discrete random variables and $\{a_1^{j_1},..., a_m^{j_m}\}$ be the set of values of $a_1,..., a_m$. Then $S_{a_1^{j_1},...,a_m^{j_m}}(t, \omega) = F(a_1^{j_1},..., a_m^{j_m}, \eta_1, \eta_2, t, \omega)$ is a random variable. The set $A_\alpha \equiv \bigcup \{\omega \mid F(a_1,..., a_m, \eta_1, \eta_2, t, \omega) \leqslant \alpha\}$ is a measurable set since A_α is the union of countably many measurable sets. For fixed t, $y(t, \omega)$ is a vector of random variables, thus there exists a sequence of discrete vector random variables $y_k(t, \omega)$ (as in [5, p. 620]) such that

$$y_k(t, \omega) \to y(t, \omega) \qquad \text{a.e.}$$

For fixed k, $F(y_k(t, \omega), \eta_1, \eta_2, t, \omega)$ is a random variable and from A1 we obtain

$$F(y_k(t, \omega), \eta_1, \eta_2, t, \omega) \to F(y(t, \omega), \eta_1, \eta_2, t, \omega) \qquad \text{a.e.,}$$

which implies that $S_y(t, \omega)$ is a random variable.

Lemma 2. *Assume that $y(t, \omega)$ belongs to L_2^m and F satisfies* A1–A4. *Then $S_y(t, \omega) = F(y(t, \omega), \eta_1, \eta_2, t, \omega)$ belongs to L_2.*

Proof. By Lemma 1 $S_y(t, \omega)$ is a random variable. From the assumptions A2–A4 we obtain

$$\|S_y(t, \omega)\|_{L_2} = \|S_y(t, \omega) - S_0(t, \omega)\|_{L_2} \leqslant \sum_{i=1}^{m} \|a_i^2(\omega)\|_{L_\infty}^{1/2} \|y_i(t, \omega)\|_{L_2} < \infty, \tag{8}$$

where

$$S_0(t, \omega) = F(0, \eta_1, \eta_2, t, \omega).$$

Lemma 3. *Assume that* A1–A4 *hold and $y(t, \omega)$ is mean square continuous with respect to t. Then $S_y(t, \omega) = F(y(t, \omega), \eta_1, \eta_2, t, \omega)$ is a mean square continuous random variable with respect to t.*

Proof. By Lemma 1, $F(y(t, \omega), \eta_1, \eta_2, t, \omega)$ is a random variable, and by Lemma 2 $F(y(t, \omega), \eta_1, \eta_2, t, \omega)$ belongs to L_2. Using assumptions A2–A4 we obtain

$$\|F(y(t+h, \omega), \eta_1, \eta_2, t+h, \omega) - F(y(t, \omega), \eta_1, \eta_2, t, \omega)\|_{L_2}$$
$$\leqslant \sum_{i=1}^{m} \|a_i^2(\omega)\|_{L_\infty}^{1/2} \|y_i(t+h, \omega) - y_i(t, \omega)\|_{L_2} + \|b(\omega)\|_{L_2} |h|, \quad (9)$$

which converges to zero as h converges to zero.

3. Existence and Asymptotic Behaviour

Using the method of [6, pp. 62–64] we shall reduce the differential equation (1) and the initial boundary conditions (2)–(5) to an integral equation. Let

$$g_1(x', t) = k \left.\frac{\partial g}{\partial x}\right|_{x=0} = \frac{x'}{t} \exp(-x'^2/4kt)/2\sqrt{\pi k t}, \quad (10)$$

where the function g has been defined by Eq. (7), and let the vector function $y(t, \omega)$ be defined by

$$y(t, \omega) = (u(x_1, y_1, z_1, t, \omega), ..., u(x_m, y_m, z_m, t, \omega)). \quad (11)$$

Proceeding as in [2] it can be shown that any function that satisfies (1)–(5) also satisfies the stochastic integral equation

$$u(x, y, z, t, \omega) = h(x, y, z, t, \omega)$$
$$+ \int_0^t \int_0^\infty \int_0^\infty g_1(x, t-\tau)\, g(y, y', t-\tau)\, g(z, z', t-\tau)$$
$$\times F(y(\tau, \omega), y', z', \tau, \omega)\, dy'\, dz'\, d\tau$$
$$+ \int_0^t \int_0^\infty \int_0^\infty g_1(y, t-\tau)\, g(x, x', t-\tau)\, g(z, z', t-\tau)$$
$$\times F(y(\tau, \omega), x', z', \tau, \omega)\, dx'\, dz'\, d\tau$$
$$+ \int_0^t \int_0^\infty \int_0^\infty g_1(z, t-\tau)\, g(x, x', t-\tau)\, g(y, y', t-\tau)$$
$$\times F(y(\tau, \omega), x', y', \tau, \omega)\, dx'\, dy'\, d\tau, \quad (12)$$

where $h(x, y, z, t, \omega)$ is defined by Eq. (6). It will always be assumed that integrals are stochastic mean square Riemann integrals [4, p. 27].

Substituting the points (x_i, y_i, z_i) $i = 1,..., m$ into Eq. (12) we obtain the vector stochastic integral equation

$$\begin{aligned} y(t, \omega) = H(t, \omega) &+ \int_0^t \int_0^\infty \int_0^\infty G_1(y', z', t-\tau) F(y(\tau, \omega), y', z', \tau, \omega) dy'\, dz'\, d\tau \\ &+ \int_0^t \int_0^\infty \int_0^\infty G_2(x', y', t-\tau) F(y(\tau, \omega), x', y', \tau, \omega)\, dx'\, dy'\, d\tau \\ &+ \int_0^t \int_0^\infty \int_0^\infty G_3(x', z', t-\tau) F(y(\tau, \omega), x', z', \tau, \omega)\, dx'\, dz'\, d\tau \\ &\equiv \mathscr{F}(y(t, \omega)), \end{aligned} \tag{13}$$

where

$$H(t, \omega) = (H_1(t, \omega),..., H_m(t, \omega)),\ H_i(t, \omega) = h(x_i, y_i, z_i, t, \omega) \tag{14}$$

and

$$G_j(\mu, \nu, t-\tau) = (G_{j1}(\mu, \nu, t-\tau),..., G_{jm}(\mu, \nu, t-\tau)), \qquad j = 1, 2, 3, \tag{15a}$$

$$\begin{aligned} G_{ji}(\mu, \nu, t-\tau) &= g_1(\delta_{1j} x_i + \delta_{2j} z_i + \delta_{3j} y_i, t-\tau) \\ &\quad \times g(\delta_{1j} y_i + \delta_{2j} x_i + \delta_{3j} x_i, \mu, t-\tau) \\ &\quad \times g(\delta_{1j} z_i + \delta_{2j} y_i + \delta_{3j} z_i, \nu, t-\tau), \qquad i = 1,..., m. \end{aligned} \tag{15b}$$

We shall use the following properties of the functions $G_{ji}(\mu, \nu, t-\tau)$, $g(x, x', \tau)$ and $g_1(x, t-\tau)$ which can be obtained directly from Eqs. (7), (10), (15)

$$G_{ji}(\mu, \nu, t-\tau) \geqslant 0, \tag{16a}$$

$$\int_0^t \int_0^\infty \int_0^\infty G_{ji}(\mu, \nu, t-\tau)\, d\mu\, d\nu\, d\tau < 1, \qquad j = 1, 2, 3, \tag{16b}$$

$$g(x, x', \tau) \geqslant 0, \tag{17a}$$

$$\int_0^\infty g(s, \tau, \sigma)\, d\tau < 1, \tag{17b}$$

$$g_1(x, t-\tau) \geqslant 0, \tag{17c}$$

$$\int_0^\infty g_1(x, \tau)\, d\tau = 1. \tag{17d}$$

We shall prove the existence of the solution of the vector integral equation (13) which implies in turn the existence of a function $u(x, y, z, t)$ satisfying the stochastic integral equation (12). It is easily seen that $u(x, y, z, t)$ satisfies Eq. (1) and conditions (2)–(5).

THEOREM 1. *Assume that F satisfies conditions* A1–A4 *and* $3m \sum_{i=1}^{m} \|a_i^2(\omega)\|_{L_\infty}^{1/2} < 1$. *Then there exists a unique solution to Eq.* (13) *belonging to C.*

Proof. We shall use Banach's fixed point theorem. Let the operator T defined on the Banach space C be given by

$$Ty(t, \omega) = \mathscr{F}(y(t, \omega)) \qquad \text{(see Eq. (13))}, \tag{18}$$

where $y(t, \omega) \in C$.

Using assumptions A2–A4, the Schwarz inequality and property (16a) we obtain the following inequalities:

$$\begin{aligned}\|Ty(t, \omega)\|_{L_2^m} \leqslant{}& \|H(t, \omega)\|_{L_2^m} \\ &+ \int_0^t \int_0^\infty \int_0^\infty \|G_2(x', y', t-\tau) F(y(\tau, \omega), x', y', \tau, \omega)\|_{L_2^m} \\ &\times dx'\, dy'\, d\tau \\ &+ \int_0^t \int_0^\infty \int_0^\infty \|G_3(x', z', t-\tau)\, F(y(\tau, \omega), x', z', \tau, \omega)\|_{L_2^m} dx'\, dz'\, d\tau \\ &+ \int_0^t \int_0^\infty \int_0^\infty \|G_1(y', z', t-\tau) F(y(\tau, \omega), y', z', \tau, \omega)\|_{L_2^m} dy'\, dz'\, d\tau\end{aligned} \tag{19}$$

and

$$\|H(t, \omega)\|_{L_2^m} = \left\{\sum_{i=1}^m EH_i^2(t, \omega)\right\}^{1/2} \leqslant \sum_{i=1}^m \{EH_i^2(t, \omega)\}^{1/2} = \sum_{i=1}^m \|H_i(t, \omega)\|_{L_2}. \tag{20}$$

Consider the second term of inequality (19):

$$\begin{aligned}&\int_0^t \int_0^\infty \int_0^\infty \|G_2(x', y', t-\tau)\, F(y(\tau, \omega), x', y', \tau, \omega)\|_{L_2^m} dx'\, dy'\, d\tau \\ &= \int_0^t \int_0^\infty \int_0^\infty \|G_2(x', y', t-\tau)\, F(y(\tau, \omega), x', y', \tau, \omega) \\ &\quad - G_2(x', y', t-\tau)\, F(0, x', y', \tau)\|_{L_2^m} dx'\, dy'\, d\tau \\ &= \int_0^t \int_0^\infty \int_0^\infty \left(\sum_{i=1}^m G_{2i}^2(x', y', t-\tau)\right)^{1/2} \|F(y(\tau, \omega), x', y', \tau, \omega) \\ &\quad - F(0, x', y', \tau, \omega)\|_{L_2} dx'\, dy'\, d\tau\end{aligned}$$

$$\leqslant \int_0^t \int_0^\infty \int_0^\infty \left(\sum_{i=1}^m G_{2i}^2(x', y', t-\tau) \right)^{1/2}$$
$$\times \left\{ E \left(\sum_{i=1}^m a_i(\omega) \, | y_i(\tau, \omega)| \right)^2 \right\}^{1/2} dx' \, dy' \, d\tau$$
$$\leqslant \int_0^t \int_0^\infty \int_0^\infty \left(\sum_{i=1}^m G_{2i}^2(x', y', t-\tau) \right)^{1/2}$$
$$\times \left\{ E \left(\sum_{i=1}^m a_i^2(\omega) \right) \left(\sum_{i=1}^m y_i^2(\tau, \omega) \right) \right\}^{1/2} dx' \, dy' \, d\tau$$
$$\leqslant \int_0^t \int_0^\infty \int_0^\infty \left(\sum_{i=1}^m G_{2i}^2(x', y', t-\tau) \right)^{1/2}$$
$$\times \left\| \sum_{i=1}^m a_i^2(\omega) \right\|_{L_\infty}^{1/2} \| y(\tau, \omega) \|_{L_2^m} \, dx' \, dy' \, d\tau$$
$$\leqslant \int_0^t \int_0^\infty \int_0^\infty \left(\sum_{i=1}^m G_{2i}(x', y', t-\tau) \right)$$
$$\times \left\| \sum_{i=1}^m a_i^2(\omega) \right\|_{L_\infty}^{1/2} \| y(\tau, \omega) \|_{L_2^m} \, dx' \, dy' \, d\tau. \tag{21}$$

In the same manner we obtain the same inequality for the third and fourth terms. Hence we obtain from (19), (20), (21)

$$\| Ty(t, \omega) \|_{L_2^m} \leqslant \sum_{i=1}^m \| H_i(t, \omega) \|_{L_2} + \int_0^t \int_0^\infty \int_0^\infty \left\| \sum_{i=1}^m a_i^2(\omega) \right\|_{L_\infty}^{1/2} \| y(\tau, \omega) \|_{L_2^m}$$
$$\times \left(\sum_{i=1}^m G_{2i}(x', y', t-\tau) \right) dx' \, dy' \, d\tau$$
$$+ \int_0^t \int_0^\infty \int_0^\infty \left\| \sum_{i=1}^m a_i^2(\omega) \right\|_{L_\infty}^{1/2} \| y(\tau, \omega) \|_{L_2^m}$$
$$\times \left(\sum_{i=1}^m G_{3i}(x', z', t-\tau) \right) dx' \, dz' \, d\tau$$
$$+ \int_0^t \int_0^\infty \int_0^\infty \left\| \sum_{i=1}^m a_i^2(\omega) \right\|_{L_\infty}^{1/2} \| y(\tau, \omega) \|_{L_2^m}$$
$$\times \left(\sum_{i=1}^m G_{1i}(y', z', t-\tau) \right) dy' \, dz' \, d\tau. \tag{22}$$

Hence,

$$\|Ty\|_C \leqslant \sum_{i=1}^{m} \sup_{t\geqslant 0} \|H_i(t,\omega)\|_{L_2} + 3m \left\| \sum_{i=1}^{m} a_i^2(\omega) \right\|_{L_\infty}^{1/2} \|y\|_C, \tag{23}$$

which implies that

$$\|Ty\|_C < \infty. \tag{24}$$

To show the continuity of $Ty(t, \omega)$, with respect to t .we consider the following inequality (as was done in (22)):

$$\begin{aligned}
&\|Ty(t+h,\omega) - Ty(t,\omega)\|_{L_2^m} \\
&\leqslant \sum_{i=1}^{m} \|H_i(t+h,\omega) - H_i(t,\omega)\|_{L_2} + \sum_{j=1}^{3} \int_0^t \int_0^\infty \int_0^\infty \left\| \sum_{i=1}^{m} a_i^2(\omega) \right\|_{L_\infty}^{1/2} \\
&\quad \times \left(\sum_{i=1}^{m} |G_{ji}(y', z', t+h-\tau) - G_{ji}(y', z', t-\tau)| \right) \cdot \|y(\tau,\omega)\|_{L_2^m} \, dy' \, dz' \, d\tau \\
&\quad + \sum_{j=1}^{3} \int_t^{t+h} \int_0^\infty \int_0^\infty \left(\sum_{i=1}^{m} G_{ji}(y', z', t+h-\tau) \right) \\
&\quad \times \left\| \sum_{i=1}^{m} a_i^2(\omega) \right\|_{L_\infty}^{1/2} \|y(\tau,\omega)\|_{L_2^m} \, dy' \, dz' \, d\tau.
\end{aligned} \tag{25}$$

Let h approach zero. The first term approaches zero by the assumption on $f(x, y, z, \omega)$ (see condition IV). The second term approaches zero by the assumption on $y(t, \omega)$, and the Lebesque dominated convergence theorem, since from (10), (15b), (17b) we obtain

$$\int_0^\infty \int_0^\infty G_{ji}(y', z', t+h-\tau) \, dy' \, dz' < g_1(a, t+h-\tau) < M, \tag{25a}$$

where

$$a = \delta_{1j} x_i + \delta_{2j} z_i + \delta_{3j} y_i, \qquad M = \frac{3\sqrt{6}\, k e^{-1.5}}{a^2 \sqrt{\pi}}.$$

It follows from the assumption on $y(t, \omega)$ and (15a) that the last term also converges to zero.

To show that T is a contraction we consider the following inequality (as done in (23))

$$\|Ty(t,\omega)-Tz(t,\omega)\|_{L_2^m}\leqslant 3m\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\|y-z\|_C \tag{26}$$

or

$$\|Ty-Tz\|_C\leqslant 3m\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\|y-z\|_C.$$

Using the assumption that $3m\,\|\sum_{i=1}^{m}a_i^2(\omega)\|_{L_\infty}^{1/2}<1$ and the Banach fixed point theorem we obtain the desired result.

To study the asymptotic behaviour of $y(t,\omega)$ for large t we consider the following theorem.

THEOREM 2. *Suppose that F satisfies* A1–A4 *and* $\|H(t,\omega)\|_{L_2^m}\to 0$ *as* $t\to\infty$. *Then* $\|y(t,\omega)\|_{L_2^m}\to 0$ *as* $t\to\infty$.

Proof. Using Eq. (13) for the solution $y(t,\omega)$ (i.e., $Ty=y$), and considering inequality (22) we obtain from (15b), $j=2$)

$$\begin{aligned}
&\int_0^t\int_0^\infty\int_0^\infty\left(\sum_{i=1}^{m}G_{2i}(x',y',t-\tau)\right)\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\|y(\tau,\omega)\|_{L_2^m}\,dx'\,dy'\,d\tau\\
&\quad=\int_0^t\int_0^\infty\int_0^\infty\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\|y(\tau,\omega)\|_{L_2^m}\\
&\qquad\times\left(\sum_{i=1}^{m}g_1(z_i,t-\tau)\,g(x_i,x',t-\tau)\,g(y_i,y',t-\tau)\right)dx'\,dy'\,d\tau\\
&\quad=\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\int_0^t\\
&\qquad\times\left\{\sum_{i=1}^{m}g_1(z_i,t-\tau)\left(\int_0^\infty g(x_i,x',t-\tau)\,dx'\right)\left(\int_0^\infty g(y_i,y',t-\tau)\,dy'\right)\right\}\\
&\qquad\times\|y(\tau,\omega)\|_{L_2^m}\,d\tau.
\end{aligned}\tag{27}$$

By (17a), (17b), (17c) we get that Eq. (27) is smaller than

$$\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\int_0^t\left\{\sum_{i=1}^{m}g_1(z_i,t-\tau)\right\}\|y(\tau,\omega)\|_{L_2^m}\,d\tau.$$

We obtain the same inequality for $j=1$, $j=3$, and hence

$$\begin{aligned}\|y(t,\omega)\|_{L_2^m} \leqslant \sum_{i=1}^{m} \|h_i(t,\omega)\|_{L_2} &+ \left\|\sum_{i=1}^{m} a_i^2(\omega)\right\|_{L_\infty}^{1/2} \int_0^t \left(\sum_{i=1}^{m} g_1(x_i, t-\tau)\right) \|y(\tau,\omega)\|_{L_2^m}\, d\tau \\ &+ \left\|\sum_{i=1}^{m} a_i^2(\omega)\right\|_{L_\infty}^{1/2} \int_0^t \left(\sum_{i=1}^{m} g_1(y_i, t-\tau)\right) \|y(\tau,\omega)\|_{L_2^m}\, d\tau \\ &+ \left\|\sum_{i=1}^{m} a_i^2(\omega)\right\|_{L_\infty}^{1/2} \int_0^t \left(\sum_{i=1}^{m} g_1(z_i, t-\tau)\right) \|y(\tau,\omega)\|_{L_2^m}\, d\tau. \end{aligned} \tag{28}$$

Using Lemma 1 of [7, p. 3], we see that

$$\begin{aligned}\|y(t,\omega)\|_{L_2^m} \leqslant \sum_{i=1}^{m} \|h_i(t,\omega)\|_{L_2} \exp \Bigg\{ &\left\|\sum_{i=1}^{m} a_i^2(\omega)\right\|_{L_\infty}^{1/2} \int_0^t \\ &\times \left[\sum_{i=1}^{m} g_1(x_i, t-\tau) + \sum_{i=1}^{m} g_1(y_i, t-\tau) + \sum_{i=1}^{m} g_1(z_i, t-\tau)\right] d\tau \Bigg\}.\end{aligned}$$

From the assumption on $H(t,\omega)$ and properties (17c), (17d) of $g_1(x, t-\tau)$ we see that for large t:

$$\|y(t,\omega)\|_{L_2^m} \leqslant O(\varepsilon) \exp \left[3m \left\|\sum_{i=1}^{m} a_i^2(\omega)\right\|_{L_\infty}^{1/2}\right]. \tag{29}$$

Hence,

$$\|y(t,\omega)\|_{L_2^m} \to 0 \quad \text{as} \quad t \to \infty.$$

Remark. If one assumes that $f(x, y, z, \omega)$ is integrable a.e. or uniformly bounded (in m.s.) on $(R^+)^3$ then one can show that $\|H(t,\omega)\|_{L_2^m} \to 0$ as $t \to \infty$ (use [5, Ch. II]).

4. The Relation between the Steady-State Solution and the Solution of the Deterministic Case

In this section we shall examine the relationship between the steady-state solution of stochastic case and the solution of the deterministic case; moreover, we shall briefly dwell upon the aging process for a system of the stochastic type.

We consider the "average problem":

Find $\bar{u} = \bar{u}(x, y, z, t)$ such that

(i) $\bar{u}_t = k(\bar{u}_{xx} + \bar{u}_{yy} + \bar{u}_{zz})$;

(ii) $\bar{u}(x, y, z, 0) = f_0(x, y, z)$;

(iii)

$$\bar{u}(0, y, z, t) = F_0(\bar{u}(x_1, y_1, z_1, t), \ldots, \bar{u}(x_m, y_m, z_m, t), y, z, t),$$

$$\bar{u}(x, 0, z, t) = F_0(\bar{u}(x_1, y_1, {}_1, t), \ldots, \bar{u}(x_m, y_m, z_m, t), x, z, t),$$

$$\bar{u}(x, y, 0, t) = F_0(\bar{u}(x_1, y_1, z_1, t), \ldots, \bar{u}(x_m, y_m, z_m, t), x, y, t),$$

where $f_0(x, y, z) = Ef(x, y, z, \omega)$.

$$F_0(\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \eta_3) = EF(\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \eta_3, \omega).$$

Let $\bar{y}_i(t) = \bar{u}(x_i, y_i, z_i, t)$ and $\bar{y}(t) = (\bar{y}_1(t), \ldots, \bar{y}_m(t))$. The non-linearity of F implies that $\bar{y}(t)$ is not equal to $Ey(t, \omega)$.

In what follows we are interested in the relation between the solution of the average problem and the solution of the stochastic problem (1)–(5) for large t. The importance of finding this relationship is that it permits us to decide whether the deterministic model (i)–(iii) is sufficiently accurate for large t.

LEMMA 4. *Assume that $\sigma^2(F(\xi, x, y, t, \omega)) \to 0$ as $t \to \infty$ for each $\xi \in R^m$ and uniformly with respect to x and y, and assume that the limit as $t \to \infty$ of $F_0(\bar{y}(t), x, y, t)$ exists uniformly with respect to x and y. Then $\sigma^2(F(\bar{y}(t), x, y, t, \omega)) \to 0$ as $t \to \infty$ uniformly with respect to x and y.*

Proof. Since we assume that the boundary terms have a limit as to $t \to \infty$, we can use the result of [1, Chap. 6], to conclude that $\bar{y}(t) \to \bar{y}(\infty)$ as $t \to \infty$. Consider

$$\begin{aligned}\sigma(F(\bar{y}(t), x, y, t, \omega)) \leqslant {} & \{E(F(\bar{y}(t), x, y, t, \omega) - F(\bar{y}(\infty), x, y, t, \omega)^2\}^{1/2} \\ & + \sigma(F(\bar{y}(\infty), x, y, t, \omega)) + |F_0(y(\infty), x, y, t) \\ & - F_0(\bar{y}(t), x, y, t)|. \qquad (30)\end{aligned}$$

By the Lipschitz continuity of F the first and the third term approach zero uniformly in x and y and it follows from the assumption of the lemma that the second term also approaches zero uniformly in x and y.

THEOREM 3. *Suppose that* A1–A4 *hold and for some M, $E_1 > 0$,*

$$\sigma^2(F(\xi, x, y, t, \omega)) \to 0 \qquad as \quad t \to \infty, \qquad (31a)$$

$$\sigma^2(F(\xi, x, y, t, \omega)) \leqslant M \qquad for \quad (\xi, x, y, t) \in R \times (R^+)^3, \qquad (31b)$$

and

$$\sigma^2(f(x, y, z)) \leqslant E. \tag{31c}$$

Then

$$\| y(t, \omega) - \bar{y}(t)\|_{L_2^m} \to 0 \qquad as \quad t \to \infty.$$

Proof. We consider the following difference

$$y(t, \omega) - \bar{y}(t) = H(t, \omega) - \overline{H(t)} + \sum_{j=1}^{3} \int_0^t \int_0^\infty \int_0^\infty (F(y(\tau, \omega), y', z', \tau, \omega) \\ - F_0(\bar{y}(\tau), y', z', \tau))\, G_j(y', z', t, \tau)\, dy'\, dz'\, d\tau. \tag{32}$$

Using the Schwarz inequality and assumptions A2–A4 we obtain

$$\begin{aligned} &\| y(t, \omega) - \bar{y}(t)\|_{L_2^m} \\ &\quad \leqslant \sum_{i=1}^{m} \| H_i(t, \omega) - \bar{H}_i(t)\|_{L_2} + \sum_{j=1}^{3} \int_0^t \int_0^\infty \int_0^\infty \left\| \sum_{i=1}^{m} a_i^2(\omega) \right\|_{L_\infty}^{1/2} \\ &\quad \times \| y(\tau, \omega) - \bar{y}(\tau)\|_{L_2^m} \cdot \left(\sum_{i=1}^{m} G_{ji}(x', y', t, \tau)\right) dx'\, dy'\, d\tau \\ &\quad + \sum_{j=1}^{3} \int_0^t \int_0^\infty \int_0^\infty \sigma(F(\bar{y}(\tau), x', y', \tau, \omega)) \\ &\quad \times \left(\sum_{i=1}^{m} G_{ji}(x', y', t, \tau)\right) dx'\, dy'\, d\tau. \end{aligned} \tag{33}$$

By assumption (31c) (see also the above remark) the first term approaches zero. The third term is equal to

$$\sum_{j=1}^{3} \int_0^{t_1} \int_0^\infty \int_0^\infty \sigma(F(\bar{y}(\tau), x', y', \tau, \omega)) \left(\sum_{i=1}^{m} G_{ji}(x'\; y', t, \tau)\right) dx'\, dy'\, d\tau \\ + \sum_{j=1}^{3} \int_{t_1}^t \int_0^\infty \int_0^\infty \sigma(F(\bar{y}(\tau), x', y', \tau, \omega)) \left(\sum_{i=1}^{m} G_{ji}(x', y', t, \tau)\right) dx'\, dy'\, d\tau. \tag{34}$$

By Lemma 4 the scond term of (34) for sufficiently large fixed t_1 is less than

$$\varepsilon \left\{ \int_{t_1}^t \sum_{i=1}^{m} (g_1(x_i, t-\tau) + g_1(y_i, t-\tau) + g_1(z_i, t-\tau))\, d\tau \right\}$$

[compare inequality (28)], and by assumption (31b) the first term is less than

$$M^{1/2}\left\{\int_0^{t_1}\left\{\sum_{i=1}^{m}\left(g_1(x_i, t-\tau)+g_1(y_i, t-\tau)+g_1(z_i, t-\tau)\right)\right\}d\tau\right\}. \quad (35)$$

Let t go to infinity. Then by (17d), (35) goes to zero. Hence the third term also approaches zero as t goes to infinity. Therefore

$$\begin{aligned}&\|y(t,\omega)-\bar{y}(t)\|_{L_2^m}\\&\quad\leqslant O(\varepsilon)+\sum_{j=1}^{3}\int_0^t\int_0^\infty\int_0^\infty\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\|y(\tau,\omega)-\bar{y}(\tau)\|_{L_2^m}\\&\qquad\times\left(\sum_{i=1}^{m}G_{ji}(x',y',t,\tau)\right)dx'\,dy'\,d\tau\end{aligned}$$

or

$$\|y(t,\omega)-\bar{y}(t)\|_{L_2^m}\leqslant O(\varepsilon)\exp\left(3m\left\|\sum_{i=1}^{m}a_i^2(\omega)\right\|_{L_\infty}^{1/2}\right)$$

([7, p. 3], compare (29)), which implies that

$$\|y(t,\omega)-\bar{y}(t)\|_{L_2^m}\to 0 \qquad \text{as} \quad t\to\infty.$$

THEOREM 4. *Assume that* $\mathrm{COV}(F(\xi,x',y',\tau,\omega),\ F(\xi,x'',y'',\sigma,\omega))=O(e^{\tau^2+\sigma^2})$, *and* $\mathrm{COV}(F(\bar{y}(\tau),y',z',\tau,\omega),\ F(\bar{y}(\sigma),y'',z'',\sigma,\omega))\leqslant M$ *for* σ, τ *belonging to any compact set in* R^+ *and* M *a positive constant. Moreover*

$$\|H(t,\omega)-\bar{H}(t)\|_{L_2^m}\to 0 \qquad \textit{as} \quad t\to\infty.$$

Then $\sup_t\|y(t,\omega)-\bar{y}(t)\|_{L_2^m}=\infty$.

Proof. Using Eq. (32) and the triangle inequality we obtain

$$\begin{aligned}&\|y(t,\omega)-\bar{y}(t)\|_{L_2^m}\\&\quad\geqslant\left\|\int_0^t\int_0^\infty\int_0^\infty\left(F(y(\tau,\omega),y',z',\tau,\omega)-F_0(\bar{y}(\tau),y',z',\tau)\right)\right.\\&\qquad\left.\times\left(\sum_{j=1}^{3}G_j(y',z',t,\tau)\right)dy'\,dz'\,d\tau\right\|_{L_2^m}-\|H(t,\omega)-\bar{H}(t)\|_{L_2^m}, \qquad (36)\end{aligned}$$

which implies that

$$\begin{aligned}
&\|y(t,\omega)-\bar{y}(t)\|_{L_2^m}\\
&\quad\geqslant \left\|\int_0^t\int_0^\infty\int_0^\infty (F(\bar{y}(\tau), y', z', \tau, \omega) - F_0(\bar{y}(\tau), y', z', \tau))\right.\\
&\qquad\times\left.\left(\sum_{j=1}^{3} G_j(y', z', t, \tau)\right) dy'\, dz'\, d\tau\right\|_{L_2^m} - \left\|\int_0^t\int_0^\infty\int_0^\infty (F(y(\tau,\omega), y', z', \tau, \omega)\right.\\
&\qquad-\left. F(\bar{y}(\tau), y', z', \tau, \omega)\left(\sum_{j=1}^{3} G_j(y', z', t, \tau)\right) dy'\, dz'\, d\tau\right\|_{L_2^m}\\
&\qquad-\|H(t,\omega)-\bar{H}(t)\|_{L_2^m}.
\end{aligned} \tag{37}$$

Hence

$$\begin{aligned}
&\|y(t,\omega)-\bar{y}(t)\|_{L_2^m} + \|H(t,\omega)-\bar{H}(t)\|_{L_2^m}\\
&\qquad+\int_0^t\int_0^\infty\int_0^\infty \|F(y(\tau,\omega), y', z', \tau, \omega) - F(\bar{y}(\tau), y', z', \tau, \omega)\|_{L_2}\\
&\qquad\times\left(\sum_{j=1}^{3}\left(\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right)\right) dy'\, dz'\, d\tau\\
&\quad\geqslant \left[\int_0^{t_1}\int_0^{t_1}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{COV}(F(\bar{y}(\tau), y', z', \tau\,\omega), F(\bar{y}(\sigma), y'', z'', \sigma, \omega))\right.\\
&\qquad\times\sum_{j=1}^{3}\left(\left(\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right)\right.\\
&\qquad\times\left(\sum_{i=1}^{m} G_{ji}(y'', z'', t, \sigma)\right) dy'\, dy''\, dz'\, dz''\, d\tau\, d\sigma\\
&\qquad+\int_{t_1}^t\int_{t_1}^t\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{COV}(F(\bar{y}(\tau), y', z', \tau, \omega), F(\bar{y}(\sigma), y'', z'', \sigma, \omega))\\
&\qquad\times\sum_{j=1}^{3}\left(\left(\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right)\right.\\
&\qquad\times\left(\sum_{i=1}^{m} G_{ji}(y'', z'', t, \sigma)\right) dy'\, dy''\, dz'\, dz''\, d\tau\, d\sigma\\
&\qquad+\int_0^{t_1}\int_{t_1}^t\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{COV}(F(\bar{y}(\tau), y', z', \tau), F(\bar{y}(\sigma), y'', z'', \sigma))\\
&\qquad\times\sum_{j=1}^{3}\left(\left(\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right)\right.
\end{aligned}$$

$$\times \left(\sum_{i=1}^{m} G_{ji}(y'', z'', t, \sigma)\right) dy'\, dy''\, dz'\, dz''\, d\tau\, d\sigma$$

$$+ \int_{t_1}^{t}\int_0^{t_1}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{COV}(F(\bar{y}(\tau), y', z', \tau), F(\bar{y}(\sigma), y'', z'', \sigma))$$

$$\times \sum_{j=1}^{3} \left(\left(\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right)\right.$$

$$\left.\times \left(\sum_{i=1}^{m} G_{ji}(y'', z'', t, \sigma)\right) dy'\, dy''\, dz'\, dz''\, d\tau\, d\sigma \right]^{1/2}. \tag{38}$$

Let t_1 be sufficiently large and fixed. By the existence of $\bar{y}(\infty)$ and the assumptions of the theorem we obtain [as in (28)]

$$\geqslant O(\varepsilon) + \left[\int_{t_1}^{t}\int_{t_1}^{t}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{COV}(F(\bar{y}(\infty), y', z', \tau, \omega),\right.$$
$$F(\bar{y}(\infty), y'', z'', \sigma, \omega))$$
$$\times \left(\sum_{j=1}^{3} \left(\left(\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right)\right.\right.$$
$$\left.\times \left(\sum_{i=1}^{m} G_{ji}(y'', z'', t, \sigma)\right) dy'\, dy''\, dz'\, dz''\, d\tau\, d\sigma\right]^{1/2}.$$

By the assumptions of the theorem the right side goes to infinity as $t \to \infty$.

Using the assumptions A2–A4 we get that the left side is smaller than [as calculated in (21)]

$$S(t) = \|y(t, \omega) - \bar{y}(t)\|_{L_2^m} + \|H(t, \omega) - \bar{H}(t)\|_{L_2^m}$$
$$+ \int_0^t\int_0^{\infty}\int_0^{\infty} \|y(\tau, \omega) - \bar{y}(\tau)\|_{L_2^m} \left\|\sum_{i=1}^{m} a_i^2(\omega)\right\|_{L_\infty}^{1/2}$$
$$\times \left(\sum_{j=1}^{3}\sum_{i=1}^{m} G_{ji}(y', z', t, \tau)\right) dy'\, dz'\, d\tau$$

and therefore $S(t)$ goes to infinity as $t \to \infty$.

Assume that $\sup_t \|y(t, w) - \bar{y}(t)\|_{L_2^m} < M$; then $S(t)$ is smaller than

$$R(t) = M + \|H(t, \omega) - \bar{H}(t, \omega)\|_{L_2^m}$$
$$+ M_1 \int_0^t\int_0^{\infty}\int_0^{\infty} \left(\sum_{j=1}^{3}\sum_{i=1}^{m} G_{ji}(y', z', t. \tau)\right) dy'\, dz'\, d\tau.$$

From (16b) and the assumption on $\|H(t, \omega) - \bar{H}(t, \omega)\|_{L_2^m}$ we get that for each t, $R(t)$ is uniformly bounded, and therefore $S(t)$ is bounded which yields a contradiction. Hence

$$\sup_t \| y(t, \omega) - \bar{y}(t)\|_{L_2^m} = \infty.$$

Discussion

We see that under the assumptions of Theorem 3 we can use the average model for large t whereas this is impossible under the condition of Theorem 4. This kind of phenomenon, where the average solution is not near the stochastic solution, is typical of the process of "aging" of the Thermoat. Its mathematical interpretation is that the variance of F converges very rapidly to infinity (i.e., the conditions of Theorem 4).

Now we consider some examples of the function $F(\xi, x, y, t, \omega)$.

(a) $F(\xi, x, y, t, \omega) = a(\omega)\, c(t)\, f(\xi, x, y)$, where $a(\omega)$ is a bounded variable $f(\xi, x, y)$ satisfies the Lipschitz condition and $f(0, x, y) = 0$. The function $c(t)$ is Lipschitz continuous bounded and as $t \to \infty$, $c(t) \to 0$. It is easily seen that F satisfies A1–A4 and, moreover, the conditions of Theorem 3 are satisfied.

(b) $F(\xi, x, y, t, \omega) = F_0(\xi, x, y, t) + \delta(\xi, \omega)$, where $F_0(\xi, x, y, t)$ is a scalar function which satisfies the Lipschitz condition and $F_0(0, x, y, t) = 0$. The function $\delta(\xi, \omega)$ is a random variable such that $\delta(0, \omega) = 0$ a.e. and satisfies a Lipschitz condition with respect to ξ. Moreover $\delta(\xi, \omega)$ satisfies

$$E(\delta(\xi, \omega)) = 0 \qquad \text{and} \qquad \sigma^2(\delta(\xi, \omega)) < \varepsilon,$$

where ε is a positive small parameter. It is easily seen that F satisfies A1–A4 and from Theorem 3 we can use the deterministic model for as ε which is small enough.

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References

1. A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice–Hall, Englewood Cliffs, N.J., 1964.
2. W. R. Mann and F. Wolf, Heat transfer between solids and gases under nonlinear boundary conditions, *Quart. Appl. Math.* **9** (1951), 163–184.

3. N. LEVINSON, Non-linear Volterra equations, *J. Math. Anal. Appl.* **1** (1960), 1–11.
4. N. U. PRABHU, "Stochastic Processes," Macmillan Co., New York, 1965.
5. J. L. DOOB, "Stochastic Processes," Wiley, New York, 1953.
6. CARSLAW AND J. JAEGER, "Conduction of Heat in Solids," Oxford Univ. Press, London/New York, 1959.
7. R. BELLMAN, "Method of Non-Linear Analysis," Vol. I, Academic Press, New York, 1970.